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Lecture 25

Measure and Integration

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$$\mathcal{A} \otimes \mathcal{B} = \mathcal{S}(\mathcal{R})$$

$$+ \mathcal{R} = \{ A \times B \mid A \in \mathcal{A}, B \in \mathcal{B} \}$$

$$\mu : \mathcal{A} \longrightarrow [0, +\infty]$$

$$\nu : \mathcal{B} \longrightarrow [0, +\infty]$$

<u>Aim</u>

$$\underline{\eta : \mathcal{A} \otimes \mathcal{B}} \longrightarrow [0, +\infty]$$

$\mathbb{R}$, $\mathcal{L}_{\mathbb{R}}$, λ — length on $\mathbb{R}$

$\mathbb{R}^2$, $E \subseteq \mathbb{R}^2$, — Area(E)

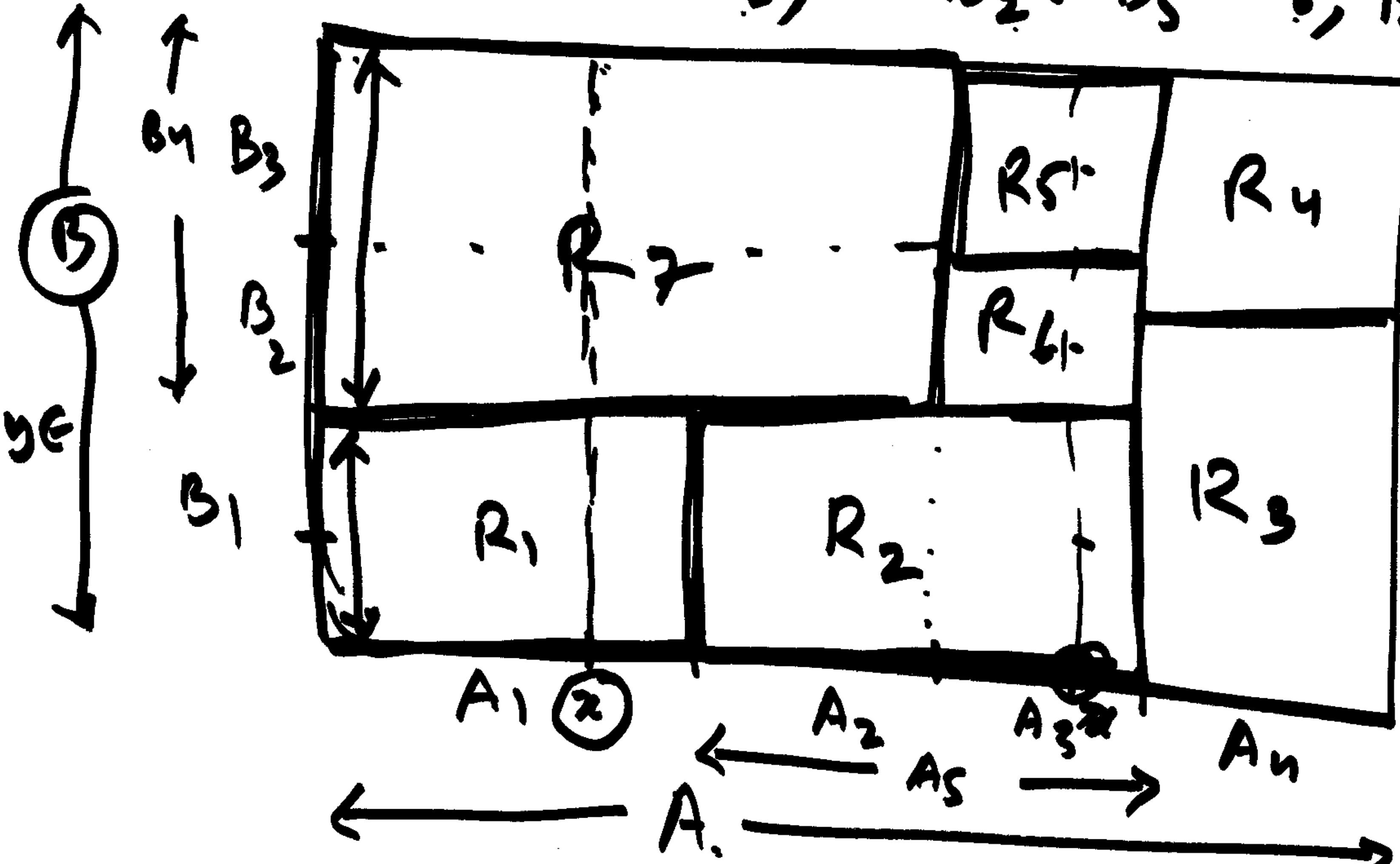
$E = I \times J$, — Area(E)

$= \underline{\lambda(I)} \cdot \underline{\lambda(J)}$

$$\overline{E \subseteq \underline{X} \times \underline{Y}}$$

$x \in A_1, y \in B_1$ or B_7 $B = B_1 \cup B_7$

$x \in A_2, y \in B_2$ or B_5 or B_6, $B = B_2 \cup B_5 \cup$



$A \times B =$

$\circledast \Longrightarrow \forall x \in A$

$$\nu(B) = \sum_{n \in S(x)} \nu(B_n) \quad \text{—} \circledast$$

$x \notin A$, then $x \notin \cancel{\ } A_n \ \forall n$

$\Longrightarrow \quad \chi_{A_n}(x) = 0.$

$x \in A$, $x \in A_n$, $n \in n(x)$, $\chi_A(x) = 1$

$$\boxed{\nu(B)\chi_A(x) = \sum_{n=1}^{\infty} \chi_{A_n}(x)\, \nu(B_n) \quad \forall x \in X}$$

$\Longrightarrow$ MC Thm on $(X, \mathcal{A}, \mu)$

$$A \times B = \bigsqcup_{n=1}^{\infty} (A_n \times B_n)$$

Fix $x \in A$, $y \in B$, then $(x,y) \in A \times B$

$\Rightarrow$ $\exists\, n$ such that $(x,y) \in A_n \times B_n$

$\Rightarrow$ $x \in A_n$, $y \in B_n$

Thus $\Rightarrow$ $y \in B \Rightarrow y \in B_n$, when $x \in A_n$

$$\Rightarrow \quad \boxed{B = \bigsqcup_{n \in S(x)} B_n} \quad (*) \checkmark$$

$$S(x) = \{ n \mid x \in A_n \}$$

$y \in B_n \cap B_m$, $n, m \in S(x)$ $\Rightarrow$ $(x,y) \in A_n \times B_n, \in A_m \times B_m$ $\otimes$

$$\int \nu(B)\chi_A(x)\,d\mu(x)$$

$$= \sum_{n=1}^{\infty} \int \chi_{A_n}(x)\,\nu(B_n)\,d\mu(x)$$

$$\nu(B)\,\mu(A) = \sum_{n=1}^{\infty} \nu(B_n)\,\mu(A_n)$$

$$\qquad \| \qquad\qquad\qquad\qquad \|$$

$$\eta(A\times B) = \sum_{n=1}^{\infty} \eta(A_n\times B_n)$$

Hence η is C. a.

$$\eta : \mathcal{A} \times \mathcal{B} \longrightarrow [0, +\infty]$$

$$\eta(A \times B) = \mu(A)\,\nu(B)$$

is a measure on semi-algebra $\underline{\mathcal{A} \times \mathcal{B}}$.

$$\xRightarrow[\text{Theory}]{\text{Extn}} \quad \tilde{\eta} : \underline{\mathcal{A} \otimes \mathcal{B}} \longrightarrow [0, +\infty]$$

$\tilde{\eta}$ a measure

$$\tilde{\eta}(A \times B) = \eta(A \times B)$$

<u>Claim</u> η is σ-finite if μ & ν are σ-finite

Note $\eta(X_i \times Y_j)$

$$= \mu(X_i)\,\nu(Y_j) < +\infty$$

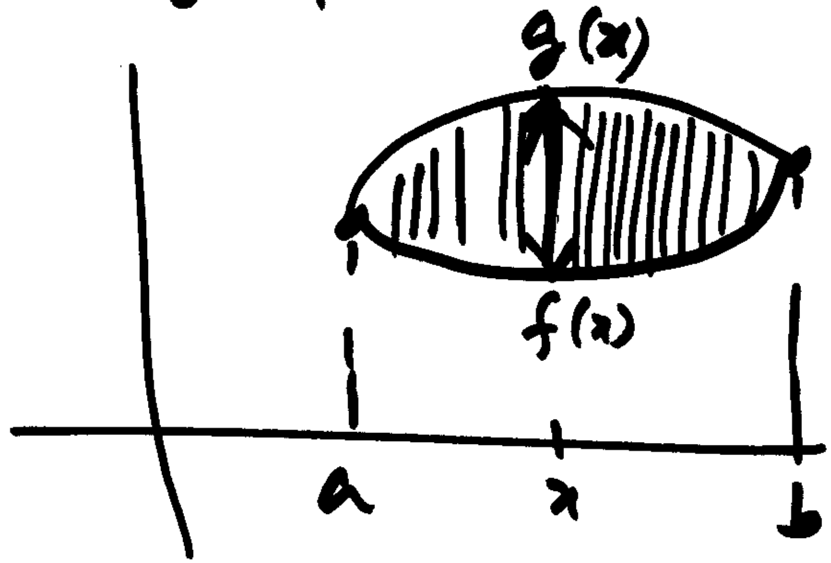
$\Rightarrow$ η is σ-finite.

$(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$

$E \subseteq X \times Y$, $E \in \mathcal{A} \otimes \mathcal{B}$.

$(\mu \times \nu)(E)$ is defined.

<u>Q</u> Can we compute $(\mu \times \nu)(E)$ using μ and ν?



$E = \{(x,y) \mid a \le x \le b,\ f(x) \le y \le g(x)\}$

$$\text{Area}(E) = \int_a^b \big(g(x) - f(x)\big)\,dx$$

$$= \int_{[a,b]} \lambda(E_x)\; d\lambda(x)$$

$$E_x = \{y : (x,y) \in E\}$$

$$= \{y \mid f(x) \leq y \leq g(x)\}$$

$E \subseteq X \times Y$

$x \in X$ fixed

$E_x = \{ y \in Y \mid (x,y) \in E \} \subseteq Y$

$\nu \cancel{\lambda} (E_x) = ? \longrightarrow E_x \in \mathcal{B} ?$

$x \longrightarrow E_x$ is $\mathcal{A}$-mbl

$\cancel{\int \lambda (E_x) \, d\lambda}$

$\int \nu (E_x) \, d\mu(x) \overset{?}{\underset{?}{=}} \eta(E)$